# REGULARIZED NUMERICAL SOLUTION OF NONLINEAR INVERSE HEAT-CONDUCTION PROBLEM

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The construction of an algorithm for a numerical solution of the nonlinear inverse problem is discussed for the case of a generalized one-dimensional heat-conduction equation in a region with moving boundaries. The algorithm is regularized in the Tikhonov manner.

In experimental studies of a variety of unsteady thermal processes, in studies of heat strength, and in various thermal tests it is necessary to construct the temperature field within the object and to determine the thermal boundary conditions from temperature measurements within the object (this is the inverse heat-conduction problem).

This problem is known to be "incorrectly formulated"; i.e., the desired results are generally not continuous functions of the input temperatures. This feature of the inverse heat-conduction problem is the main difficulty involved in solving it; as a result of this feature, direct methods do not give stable approximations of the desired functions with small time steps in the integration [1]. If the time steps are made larger, the results may become very inaccurate.

Regularized algorithms for solving linear, one-dimensional, inverse heat-conduction problems, which impose no restrictions on the size of the time integration step, were worked out in [2, 3]. If the thermal processes are very intense and last for a long time, the formulation of the inverse problems generally becomes very complicated. It becomes necessary to take into account the changes in the thermal properties of the material with the temperature, and in several cases it is necessary to choose a heat-conduction model with internal heat and mass sources and with the motion of a liquid or gas through pores in the object. Here the boundaries of the object can move during the heating. A typical example of this model is the operation of a composite heat-shielding material under conditions of heat damage in the interior and removal of material.

In the present paper we propose a method for obtaining stable solutions of the nonlinear inverse problem for a generalized one-dimensional heat-conduction equation corresponding to heat transfer in a porous object with internal heat and mass evolution. The problem is treated in a region with moving boundaries, described by the functions $X_1(t)$ and $X_2(t)$. The boundary motion can be caused by a linear removal of material or by thermal or mechanical deformation of the object. This method is based on a regularization by the Tikhonov method of an implicit difference scheme for integrating the heat-conduction equation along the direction of the spatial coordinate [4].

Mathematically, the problem is formulated as the problem of the spatial continuation of the solution of the heat-conduction equation, from the boundary $X_2(t)$, at which Cauchy's conditions are specified [the heat flux $q_2(t)$ and the temperature $f(t)$], to the boundary $X_1(t)$. If the temperature is measured within the object, and if some boundary condition is known at the right-hand end of the region, then it is first necessary to solve the boundary-value problem in the region between the point at which the measurements are made and the right-hand boundary. Then we can obtain the formulation of the inverse heat-conduction problem with which we are concerned [4].

Our problem is thus to determine the function $T_W(t)$ from the conditions

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$$C(t)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)+k(T)\frac{\partial T}{\partial x}+\psi(T), \tag{1}$$

$$X_1(t)<x<X_2(t),\ 0<t\leqslant t_m,$$

$$T(x,\ 0)=\varphi(x),\ X_1(0)\leqslant x\leqslant X_2(0), \tag{2}$$

$$T(X_1(t),\ t)=T_w(t), \tag{3}$$

$$T(X_2(t),\ t)=f(t), \tag{4}$$

$$-\lambda(T(X_2(t),\ t))\frac{\partial T(X_2(t),\ t)}{\partial x}=q_2(t). \tag{5}$$

Introducing the new variables

$$\xi=\frac{x-X_1(t)}{X_2(t)-X_1(t)},\ \tau=t, \tag{6}$$

we transform to a rectangular integration region [5]. Then problem (1)-(5) transforms to

$$C(T)\frac{\partial T}{\partial\tau}=\frac{1}{[X_2(\tau)-X_1(\tau)]^2}\left[\lambda(T)\frac{\partial^2 T}{\partial\xi^2}+\frac{\partial\lambda}{\partial\xi}\cdot\frac{\partial T}{\partial\xi}\right]+$$
$$+\frac{C(T)[\dot{X}_1(\tau)-\xi(\dot{X}_2(\tau)-\dot{X}_1(\tau))]+k(T)}{X_2(\tau)-X_1(\tau)}\frac{\partial T}{\partial\xi}+\psi(T),$$
$$0<\xi<1,\ 0<\tau\leqslant\tau_m, \tag{7}$$

$$T(\xi,\ 0)=\varphi(\xi),\ 0\leqslant\xi\leqslant 1, \tag{8}$$

$$T(0,\ \tau)=T_w(\tau), \tag{9}$$

$$T(1,\ \tau)=f(\tau), \tag{10}$$

$$-\frac{\lambda(T(1,\ \tau))}{X_2(\tau)-X_1(\tau)}\frac{\partial T(1,\ \tau)}{\partial\xi}=q_2(\tau). \tag{11}$$

We introduce the difference grid $\omega=\{\xi_i=hi,\ i=0,\ 1,\ \ldots,\ n;\ \tau_j=\Delta\tau j,\ j=0,\ 1,\ \ldots,\ m\}$. Alifanov et al. [4] have shown that when an implicit approximation scheme is used it is necessary to solve, at each i-th spatial layer ($i=n-2,\ n-3,\ \ldots,\ 0$), a system of nonlinear algebraic equations:

$$T_i^0=\varphi_i,$$
$$A_i^jT_i^{j+1}+B_i^jT_i^j+D_i^jT_i^{j-1}=F_i^j,\ j=1,\ 2,\ \ldots,\ m-1, \tag{12}$$
$$T_i^m=\varkappa_i^mT_i^{m-1}+\nu_i^m,$$

where

$$A_i^j=\frac{C_i^j}{2\Delta\tau};\quad B_i^j=\frac{C_i^j[\dot{X}_1^j+\xi_i(\dot{X}_2^j-\dot{X}_1^j)]+k_i^j}{h(X_2^j-X_1^j)}+\frac{\lambda_{i+1}^j-2\lambda_i^j}{h^2(X_2^j-X_1^j)^2};$$

$$D_i^j=-\frac{C_i^j}{2\Delta\tau};\quad F_i^j=\frac{\lambda_i^jT_{i+2}^j-3(\lambda_i^j-\lambda_{i+1}^j)T_{i+1}^j}{h(X_2^j-X_1^j)^2}+\frac{C_i^j[\dot{X}_1^j+\xi_i(\dot{X}_2^j-\dot{X}_1^j)]+k_i^j}{h(X_2^j-X_1^j)}T_{i+1}^j+\psi_i^j.$$

Here $\varkappa_i^m$ and $\nu_i^m$ are governed by the condition imposed on the solution at $\tau=\tau_m$. For example, with an a priori specification of the second derivative, i.e., $[\partial^2T_i(\tau_m)/\partial\tau^2]=C_2=\text{const}$, $i=n-1,\ n-2,\ \ldots,\ 0$, it can be shown that we have

$$\varkappa_i^m=\frac{A_i^m-D_i^m}{2A_i^m+B_i^m},\quad \nu_i^m=\frac{F_i^m-A_i^mC_2\Delta\tau^2}{2A_i^m+B_i^m}, \tag{13}$$

where $A_i^m$, $B_i^m$, $D_i^m$, $F_i^m$ are determined from Eqs. (12) with $j=m$.

For small time steps $\Delta\tau$ and if the function $f(\tau)$ is specified with some error, the "conditionality" of the matrix of system (12) is poor, and the problem of determining the solution of the system is an incorrect problem in the Hadamard sense [6].

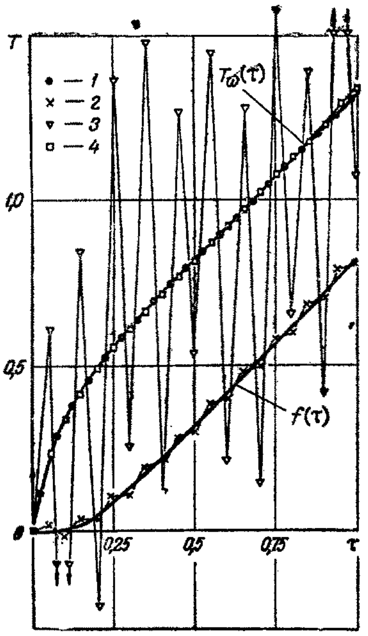


Fig. 1. Results of the solution of the model problem. Solid curve) exact numerical problem. 1) $T_W(\tau)$ according to the exact input temperatures with $\alpha = 0$; 2) perturbed input data, $\Delta f_j = \pm 0.05 f_{max}$, $j = 1, 2, \ldots, m$; 3) $T_W(\tau)$ according to the perturbed data with $\alpha = 0$; 4) $T_W(\tau)$ obtained by the regularization method with $\alpha = \alpha_{c.0}$.

To construct a stable algorithm for solving problem (12), we can use the regularization method of [7]. We write the Tikhonov functional in the form

$$\Phi_i(\alpha) = \sum_{j=2}^{m-1} (A_i^j T_i^{j+1} + B_i^j T_i^j + D_i^j T_i^{j-1} - F_i^j)^2 + (A_i^1 T_i^2 + B_i^1 T_i^1 - F_i^1 + D_i^1 \varphi_i)^2 + (T_i^m - \varkappa_i^m T_i^{m-1} - \nu_i^m)^2 + \frac{\alpha k_1}{\Delta\tau^2} \sum_{j=1}^{m} (T_i^j - T_i^{j-1})^2 + \frac{\alpha k_2}{\Delta\tau^4} \sum_{j=1}^{m} (T_i^{j+1} - 2T_i^j + T_i^{j-1})^2, \quad (14)$$

where $\alpha$ is the regularization parameter, and $k_1 > 0$ and $k_2 > 0$ are certain nonnegative numbers.

Minimizing (14) with respect to all the $T_i^j$, $j = 1, 2, \ldots, m$, we find a system of nonlinear algebraic equations with a symmetric, five-diagonal, positive-definite matrix:

$$\sum_{k=j-2}^{j+2} a_{j,k} T_i^j = b_j, \quad j = 1, 2, \ldots, m, \quad (15)$$

where

$$a_{j,j} = \begin{cases} (B_i^1)^2 + (D_i^2)^2 + \alpha\left(\frac{2k_1}{\Delta\tau^2} + \frac{5k_2}{\Delta\tau^4}\right), & j = 1, \\ (A_i^{j-1})^2 + (B_i^j)^2 + (D_i^{j+1})^2 + \alpha\left(\frac{2k_1}{\Delta\tau^2} + \frac{6k_2}{\Delta\tau^4}\right), & j = 2, 3, \ldots, m-2, \\ (A_i^{m-2})^2 + (B_i^{m-1})^2 + (\varkappa_i^m)^2 + \alpha\left(\frac{2k_1}{\Delta\tau^2} + \frac{5k_2}{\Delta\tau^4}\right), & j = m-1, \\ A_i^{m-1} + 1 + \alpha\left(\frac{k_1}{\Delta\tau^2} + \frac{k_2}{\Delta\tau^4}\right), & j = m; \end{cases}$$

$$a_{j,j+1} = a_{j+1,j} = \begin{cases} A_i^j B_i^j + B_i^{j+1} D_i^{j+1} - \alpha\left(\frac{k_1}{\Delta\tau^2} + 4\frac{k_2}{\Delta\tau^4}\right), & j = 1, 2, \ldots, m-2, \\ A_i^{m-1} B_i^{m-1} - \varkappa_i^m - \alpha\left(\frac{k_1}{\Delta\tau^2} + 2\frac{k_2}{\Delta\tau^4}\right), & j = m-1, \\ 0, & j = m; \end{cases}$$

$$a_{j,j+1} = a_{j+2} = \begin{cases} A_i^{j+1} D_i^{j+1} + \alpha\frac{k_2}{\Delta\tau^4}, & j = 1, 2, \ldots, m-2, \\ 0, & j = m-1, m; \end{cases}$$

$$b_1 = F_i^1 B_i^1 + F_i^2 D_i^2 - B_i^1 D_i^1 \varphi_i + \alpha\varphi_i\left(\frac{k_1}{\Delta\tau^2} + 2\frac{k_2}{\Delta\tau^4}\right);$$

$$b_2 = F_i^1 A_i^1 + F_i^2 B_i^2 + F_i^3 D_i^3 - \alpha\varphi_i\frac{k_2}{\Delta\tau^4};$$

$$b_j = F_i^{j-1} A_i^{j-1} - F_i^j B_i^j + F_i^{j+1} D_i^{j+1}, \quad j = 3, 4, \ldots, m-2;$$

$$b_{m-1} = F_i^{m-2} A_i^{m-2} + F_i^{m-1} B_i^{m-1} - \varkappa_i^m \nu_i^m - \alpha C_2 \frac{k_2}{\Delta\tau^2};$$

$$b_m = F_i^{m-1} A_i^{m-1} + \nu_i^m + 2\alpha C_2 \frac{k_2}{\Delta\tau^2}.$$

Here the temperature $T_i^0 = \varphi_i$ is assumed known quite accurately, since in a real experiment it is frequently possible to arrange a constant initial temperature distribution and to determine it accurately.

The solution of system (15) yields the regularized desired function for a fixed parameter $\alpha$. Problem (15) should be solved by the square-root method.

The optimum approximation can be chosen by the quasioptimum-parameter method of [8] ($\alpha_{l+1} = \varkappa\alpha_l$, $\varkappa > 0$). Here we choose an effective value of the parameter which is constant for all i, i = n − 2, n − 3, ..., 0:

$$\min_l \{\max_j |T^j_{w,l+1} - T^j_{w,l}|\}. \tag{16}$$

If we know the error with which the input data are specified,

$$\delta = \Big[\sum_{j=1}^{m} \sigma_j^2\Big]^{\frac{1}{2}},$$

where $\sigma_j$ is the mean square error of the function $f(\tau)$ for $\tau = \tau_j$, then a better approximation can be found by using the discrepancy principle of [9]:

$$\Big[\sum_{j=1}^{m} (T^j_n - f_j)^2\Big]^{\frac{1}{2}} - \delta = 0. \tag{17}$$

Here $T^j_n$, j = 1, 2, ..., m, is the solution of the direct heat-conduction problem in the region $\{\xi_i \le \xi \le 1, 0 \le \tau \le \tau_m\}$ under condition (11) at the right-hand boundary of the object and with a known temperature $T^j_i$, j = 1, 2, ..., m, at the left-hand end of the region at $\xi = \xi_i$. The values of $T^j_i$ are found by solving system (15).

Using condition (17), we can either choose the regularization parameter at each spatial step or choose an effective value of the parameter for the entire region or for the various parts of the region.

Condition (17) can be used to construct an algorithm for automatically seeking the parameter $\alpha$. For this purpose we consider, instead of Eq. (17), the relation

$$F(p) = \Big[\sum_{j=1}^{m} (T^j_n - f_j)^2\Big]^{-\frac{1}{2}} - \delta^{-1} = 0, \quad p = \frac{1}{\alpha}, \tag{18}$$

for which we use the Newton iterative method [10]

$$p_{k+1} = p_k - \frac{F(p_k)}{F'(p_k)}, \tag{19}$$

where the derivative F'(p) is determined numerically:

$$F'(p) = -\frac{1}{p^2}\,\frac{F(\alpha + \Delta\alpha) - F(\alpha - \Delta\alpha)}{2\Delta\alpha}.$$

This algorithm for solving the inverse problem has been incorporated in an ALGOL program for a BÉSM-6 computer. The results calculated for one model example are shown in Fig. 1. We treated the problem of determining the temperature of the outer surface of an unbounded plate of thickness b = 1 with constant thermal properties:

$$C(T) = \lambda(T) = 1, \quad k(T) \equiv \psi(T) = 0.$$

The plate with fixed boundaries is heated at its outer side by heat conduction at a rate $q_1(\tau) = 1$; the other surface ($q_2 = 0$) is insulated [11].

The temperature at the outer surface, $T_w(\tau)$, reconstructed on the basis of exact input data (grid of n × m = 50 × 50, $\Delta Fo = (\lambda\Delta\tau/cb^2) = 0.02$) is stable, even with $\alpha = 0$, when the algorithm constitutes a direct solution method. When a sawtooth perturbation is imposed on the input data ($\Delta f = \pm 5\%\ f_{max}$), a clearly defined instability appears in the calculation. A stable solution of this problem was found by the method of the quasioptimum regularization parameter. In this case condition (16) was satisfied with $\alpha$ = 0.86.

## NOTATION

C(T), specific heat at constant volume; $\lambda$(T), thermal conductivity; k(T), filtration coefficient; $\psi$(T), distributed heat source (or sink); T, temperature; x, $\xi$, coordinate; t, $\tau$, time; h, integration step along the spatial coordinate; $\Delta\tau$, integration step along the time; $\varphi$(x) initial temperature distribution; q, heat flux; X, coordinate of boundary; f, input data; $\delta$, error in the specification of the input temperatures; $T_W(\tau)$, temperature of outer surface; $\tau_m$, value of the time interval at the right-hand boundary; $\alpha$, regularization parameter.

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